



THE UNIVERSITY *of York*

*Discussion Papers in Economics*

No. 11/17

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With Markovian Payoffs**

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# Equilibria in Continuous Time Preemption Games with Markovian Payoffs\*

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September 8, 2011

## Abstract

This paper studies timing games in continuous time where payoffs are stochastic and strongly Markovian. The main interest is in characterizing equilibria where players preempt each other along almost every sample path. It is found that the existence of such preemption equilibria depends crucially on whether there is a coordination mechanism that allows for rent equalization or not, and whether the stochastic payoffs admit upward jumps. Through numerical examples it is argued that the possibility of such coordination improves social welfare and that the welfare loss due to preemption decreases in uncertainty.

*Keywords:* Timing Games, Real Options, Preemption

*JEL classification:* C73, D43, D81

## 1 Introduction

In many competitive timing situations the first mover has an advantage: the firm that first adopts a new technology, the first developer of a new real-estate opportunity, etc. However, if the leader role is not exogenously determined, then the competition to become the leader may erode that first mover advantage. Indeed, the standard prediction from the literature on timing games with a first mover advantage is that preemption equalizes the expected payoffs of the first and second mover. This point has been made ever since such early contributions as, for example, Posner (1975).

Preemption is often analyzed in a continuous time framework. This can lead to a coordination problem, because in continuous time it is impossible to distinguish

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\*The author gratefully acknowledges helpful comments from seminar participants at the University of York and the 2011 SAET conference in Faro, Portugal.

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between simultaneous action and immediate reaction. Fudenberg and Tirole (1985) solve this problem by using a technique from optimal control which allows them to show that in two-player deterministic timing games with a first mover advantage there always exists an equilibrium in which rents are equalized. Unfortunately, their equilibrium concept is rather complicated as it involves players to choose a distribution function describing the probability with which they act before any point in time, as well as an “intensity over an interval of atoms” when players wish to act at the same time. It is this device that allows the derivation of a preemption equilibrium where (i) players do not act simultaneously when this is not optimal (an outcome they refer to as a “coordination failure”) and (ii) symmetric players each act first with probability  $1/2$  at the preemption point.

In most real-life preemptive situations the future is not known with certainty and, therefore, a deterministic timing game may not be the best modeling tool. Using techniques from optimal stopping theory, several papers have studied timing games in which players’ payoffs are subject to random shocks.<sup>1</sup> This introduces an “option value of waiting” into the payoffs which, in general, delays stopping. As can be imagined, the addition of uncertainty complicates the game theoretic analysis even further. As in the deterministic case this is mainly due to the difficulty of solving the coordination problem that arises when it is a best response for both players to stop while these are only best responses if only one player succeeds. In the literature this is often solved by making fairly ad-hoc assumptions based on Fudenberg and Tirole (1985). It is not at all clear, however, that this is appropriate. In addition, this approach could not deal with asymmetries. Thijssen et al. (2002) extend the Fudenberg and Tirole (1985) concepts in an appropriate way to a stochastic setting. They show that, qualitatively, the equilibria are not changed by the introduction of uncertainty. It is not clear, however, how restrictive their assumptions are. In addition, the level of technicality required to derive the results is such that it is not readily applicable.

In this paper I take a different approach to the problem of preemption in a real options model. Its contribution is three-fold. First, the analysis presented here separates, as much as possible, the timing and coordination issues involved in preemption models. This makes it easier to prove equilibrium existence and also makes it clearer how the different assumptions needed to guarantee existence of equilibrium interact. The simplicity of the arguments is based on an exploitation of the strong Markovian nature of the underlying stochastic process, which allows one to take the range of the process as the state space, rather than time itself. As a result, this paper presents

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<sup>1</sup>See, for example, Boyer et al. (2004), Pawlina and Kort (2006, 2010), Bouis et al. (2009), Roques and Savva (2009), Mason and Weeds (2010), and Thijssen (2010).

a more complete picture of all equilibria that can exist in preemption games. Second, this paper starts from fairly general assumptions on primitives, which enables one to see how economic primitives interact with equilibrium existence. This also allows for a comprehensive analysis of asymmetric games. Finally, the paper allows for a large class of underlying stochastic processes. The existing literature almost exclusively relies on geometric Brownian motion. This paper allows also for an analysis of other processes such as mean-reverting diffusions and Lévy processes with downward jumps. Numerical illustrations show that the nature of the underlying stochastic process has no influence on the qualitative nature of equilibrium, but can have a substantial impact on estimates of, for example, values of waiting and welfare costs of preemption.

The paper is organized as follows. The basic ingredients of the model are described in Section 2. Section 3 discusses the case where there is no competition for the leader role, i.e. the standard Stackelberg approach. In this case, no coordination problem arises. The model is still useful, however, to provide a contrast with the case of endogenous roles, which is dealt with in Section 4. That section introduces the main strategy and equilibrium concepts, as well as the equilibrium results regarding preemptive equilibria in the spirit of Fudenberg and Tirole (1985). The arguments in Section 4 are kept as simple as possible by assuming that the coordination problem is solved exogenously. In particular, the assumption of rent-equalization turns out to be crucial for the existence of equilibria in which preemption takes place. Attention is also paid to equilibria that do not lead to preemptive behaviour. A non-cooperative defense of why the assumption of rent-equalization is reasonable is given in Section 5. Implications of the theory for predictions on preemptive behavior under different stochastic processes are presented in Section 6. Numerical examples show that, both for spectrally negative exponential jump-diffusions and exponential mean-reverting diffusions the preemption region is increasing in volatility. This indicates that preemptive behavior can be expected to be observed more often in situations with higher levels of uncertainty. This section also shows how the theory can be used to make welfare predictions in a model of industry investment. In particular, it is illustrated how exogenous versus endogenous firm roles can lead to different welfare losses. It is found, for example, that (under geometric Brownian motion) competition for the leader role always leads to a higher social welfare than a non-competitive situation. It is also shown, however, that the preemptive equilibrium does not lead to a social optimum. Interestingly, the welfare loss in both cases decreases as volatility increases. Section 7, finally, concludes.

## 2 The Model

Consider a situation where two players  $i \in \{1, 2\}$  have to decide on a stopping time over an infinite time horizon. Their payoffs are influenced by a state-variable which takes values in  $E = (a, b) \subseteq \mathbb{R}$ . Let  $\bar{E}$  denote the closure of  $E$  (in the standard topology on  $\mathbb{R}$ ). For each  $y \in E$ , the state variable follows a strong Markovian càdlàg (right-continuous with left-limits) semimartingale  $(Y_t)_{t \geq 0}$  on a probability space  $(\Omega, \mathcal{F}, P_y)$ , endowed with a filtration  $(\mathcal{F}_t)_{t \geq 0}$ , with  $Y_0 = y$ ,  $P_y$ -a.s. The process  $(Y_t)_{t \geq 0}$  is assumed to be adapted to  $(\mathcal{F}_t)_{t \geq 0}$ .

It is assumed that both players discount payoffs at the constant and common rate  $r > 0$ . All strategies in this paper take the form of *hitting times*. These are stopping times of the form  $\tau(Y^*) := \inf\{t \geq 0 | Y_t \geq Y^*\}$ , for some  $Y^* \in \bar{E}$ , where  $\tau_y(a) = 0$  and  $\tau_y(b) = \infty$ ,  $P_y$ -a.s, for all  $y \in E$ . For  $y \in E$  and  $Y^* \in \bar{E}$ , let  $\nu_y(Y^*)$  denote the Laplace transform of  $\tau(Y^*)$  (under  $P_y$ ) evaluated at  $r$ , i.e.

$$\nu_y(Y^*) := \mathbb{E}^{P_y} \left[ e^{-r\tau(Y^*)} \right].$$

Note that  $\nu_y(Y^*) = 1$  for all  $y \geq Y^*$ , that  $\nu_y(b) = 0$ , for all  $y \in E$ , and that, because of the strong Markov property, it holds that  $\nu_y(Y^2) = \nu_y(Y^1)\nu_{Y^1}(Y^2)$ , for all  $y < Y^1 < Y^2$ .

The following assumption is made on the stochastic environment.

**Assumption 1.** The process  $(Y_t)_{t \geq 0}$  has no upward jumps and is such that for all  $y \in E$  the function  $\nu_y(\cdot)$  is continuous.

This assumption allows for many underlying processes, like arithmetic Brownian motion, geometric Brownian motion, mean-reverting diffusions, and spectrally negative Lévy processes. The requirement that  $(Y_t)_{t \geq 0}$  has no upward jumps is equivalent to saying that the supremum process  $(\bar{Y}_t)_{t \geq 0}$ , defined by  $\bar{Y}_t = \sup_{0 \leq s \leq t} Y_s$ , has continuous sample paths. This assumption simplifies finding solutions to the optimal stopping problems below considerably. It also often makes deriving  $\nu_y(\cdot)$  fairly straightforward (see Section 6 for some examples). On the other hand, it limits the number of types of behavior that are consistent with equilibrium.<sup>2</sup>

The payoffs accruing to the players depend on their “stopping status”  $k \in \{0, 1\}$ , which indicates whether a player has stopped ( $k = 1$ ) or not ( $k = 0$ ). Let  $D_{k\ell}^i(y)$ ,  $y \in \bar{E}$ , denote the expected present value of stopping (under  $P_y$ ) to Player  $i$  if her stopping status is  $k$ , the stopping status of Player  $j$ ,  $j \neq i$ , is  $\ell$ , and the state variable has value  $y$ . In addition, it is assumed that stopping entails incurring a once off sunk cost  $I^i > 0$ .

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<sup>2</sup>See Boyarchenko and Levendorskiĭ (2011) for an analysis with positive jumps.

**Assumption 2.** For all  $i \in \{1, 2\}$ , it holds that

1.  $D_{k\ell}^i$ ,  $k, \ell = 0, 1$ , is continuous on  $E$ ;
2.  $D_{10}^i(a) - D_{00}^i(a) < I^i$  and  $D_{10}^i(b) - D_{00}^i(b) > I^i$ ;
3.  $D_{10}^i(\cdot) - D_{00}^i(\cdot)$  is strictly increasing;
4.  $D_{11}^i(\cdot) - D_{01}^i(\cdot)$  non-decreasing;
5.  $D_{10}^i(\cdot) - D_{00}^i(\cdot) > D_{11}^i(\cdot) - D_{01}^i(\cdot) \geq D_{11}^i(\cdot) - D_{00}^i(\cdot) \geq 0$ .

Assumptions 2.2 and 2.3 ensure that there is a unique threshold for the state variable where the net present value of being the first player to stop becomes positive. Assumption 2.4 allows for the possibility that it is never optimal to become the second mover. Assumption 2.5 implies that there is a first-mover advantage and that  $D_{10}^i(\cdot) > D_{11}^i(\cdot) \geq D_{00}^i(\cdot) \geq D_{01}^i(\cdot)$ .

The final assumption that is made ensures that waiting forever renders the option valueless. This assumption essentially rules out speculative bubbles.

**Assumption 3.** The functions  $\nu_y(\cdot)$ , and  $D_{k\ell}^i(\cdot)$  are such that  $\lim_{Y^* \uparrow b} \nu_y(Y^*)[D_{1k}^i(Y^*) - D_{1\ell}^i(Y^*) - I^i] = 0$ , for all  $k \geq \ell$ .

### 3 Exogenous Leader and Follower Role

Before analyzing the game where players vie for the leader role, let's first study the standard Stackelberg model applied to a situation where players have to choose stopping times. Assume that Player  $i$  is the leader in this game. Player  $j$ , hence, is the follower. The strategies in this game are going to be the thresholds at which the players exercise their options. We want to allow for the possibilities that Player  $i$  stops immediately, no matter what the value of the state variable, and that Player  $j$  never stops. So, the strategy space is taken to be  $\bar{E}$ . For a pair of strategies  $(Y^i, Y^j)$ , with  $Y^i \leq Y^j$ , the payoff to Player  $i$  is the *leader value*, which, for all  $y \in E$ , is equal to

$$\begin{aligned} L_y^i(Y^i; Y^j) := & D_{00}^i(y) + \nu_y(Y^i)[D_{10}^i(Y^i) - D_{00}^i(Y^i) - I^i] \\ & + \nu_y(Y^j)[D_{11}^i(Y^j) - D_{10}^i(Y^j)]. \end{aligned} \quad (1)$$

The expected payoff to the follower equals

$$\begin{aligned} F_y^j(Y^j; Y^i) = & D_{00}^j(y) + \nu_y(Y^i)[D_{01}^j(Y^i) - D_{00}^j(Y^i)] \\ & + \nu_y(Y^j) \left( D_{11}^j(Y^j) - D_{01}^j(Y^j) - I^j \right). \end{aligned} \quad (2)$$

This formulation depends crucially on the strong Markovian nature of  $(Y_t)_{t \geq 0}$  and on the assumption that there are no positive jumps. This assumption, namely, guarantees that one can write

$$\begin{aligned} \mathbb{E}_y \left[ e^{-r\tau(Y^*)} (D_{1k}^i(Y_{\tau(Y^*)}) - D_{0k}^i(Y_{\tau(Y^*)}) - I^i) \right] \\ = \mathbb{E}_y \left[ e^{-r\tau(Y^*)} \right] (D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i), \end{aligned}$$

for all  $Y^* \in \bar{E}$ .

A *subgame perfect Markov equilibrium* (SPME) is a pair of strategies  $(\bar{Y}^i, \bar{Y}^j) \in \bar{E} \times \bar{E}$ , such that for all  $(Y^i, Y^j) \in \bar{E} \times \bar{E}$  it holds that

$$L_y^i(\bar{Y}^i; \bar{Y}^j) \geq L_y^i(Y^i; Y^j), \quad \text{and} \quad F_y^j(\bar{Y}^j; \bar{Y}^i) \geq F_y^j(Y^j; Y^i).$$

The following lemma describes the solutions to the maximization problems faced by the players.

**Lemma 1.** *Suppose that Assumptions 1–3 hold. The functions  $L(\cdot)$  and  $F(\cdot)$  have unique maximizers  $Y_L^i \in E$  and  $Y_F^j \in \bar{E}$ , respectively, which satisfy  $Y_L^i \leq Y_F^j$ .*

**Proof.** Existence of  $Y_L^i$  and  $Y_F^j$  follows trivially from the continuity of  $D_{kl}^i(\cdot)$  and  $\nu_y(\cdot)$ , and the fact that  $\bar{E}$  is a compact set. Due to Assumptions 2.2, 2.3, and 3,  $L_y^i(\cdot)$  attains its maximum on  $E$ . The fact that  $Y_L^i \leq Y_F^j$  follows from Assumptions 2.3–5.

Uniqueness of  $Y_F^j$  is established as follows. Define  $g^j(\cdot) := D_{1k}^j(\cdot) - D_{1\ell}^j(\cdot) - I^j$ . Note that the maximizers of  $F_y^j(\cdot)$  are the same as the maximizers of  $g^j(\cdot)$ . Suppose that  $Y_1$  and  $Y_2$  are two distinct maximizers of  $g^j$ , such that (wlog)  $Y_1 < Y_2$ . First, assume that  $Y_2 < b$ . Then it holds that  $g^j(Y_1) = \nu_{Y_1}(Y_2)g^j(Y_2)$ , and, thus, that  $g^j(Y_2) > g^j(Y_1)$ . Continuity of  $g^j(\cdot)$  implies that there exists  $Y_3 \in (Y_1, Y_2)$ , such that  $g^j(Y_3) > g^j(Y_1)$ . Therefore, it holds that

$$\begin{aligned} \nu_{Y_1}(Y_3)\nu_{Y_3}(Y_2) &= \nu_{Y_1}(Y_2) = \frac{g^j(Y_1)}{g^j(Y_2)} < \frac{g^j(Y_3)}{g^j(Y_2)} \\ &\stackrel{*}{\iff} \nu_{Y_3}(Y_2)g^j(Y_2) < \nu_{Y_1}(Y_3)g^j(Y_3) \\ &\iff \nu_{Y_1}(Y_2)g^j(Y_2) = g^j(Y_1) < \nu_{Y_1}(Y_3)g^j(Y_3), \end{aligned}$$

where  $(*)$  follows from  $g^j(Y_2) > 0$ , since  $Y_2 < b$ . But this is a contradiction to  $Y_1$  being a maximizer.

Finally, if  $Y_1 = b$  is a maximizer, uniqueness follows from the fact that there is no  $Y^*$  for which  $g^j(Y^*) > 0$ . After all, if there were, then

$$\nu_y(Y^*)g^j(Y^*) > \nu_y(b)g^j(b) = 0.$$

A similar reasoning shows uniqueness of  $Y_L^i$ . ■

The following proposition describes the unique asymmetric SPME.



**Proposition 1.** *The unique asymmetric SPME is  $(Y_L^i, Y_F^j)$ .*

The proof of this proposition follows trivially from the uniqueness of  $Y_F^j$  and  $Y_L^i$ . This asymmetric equilibrium leads to players stopping sequentially. Such equilibria are referred to in the remainder as *sequential stopping equilibria*.

Symmetric equilibria may also exist. To derive these, note that, if players choose symmetric strategies  $(Y^*, Y^*)$ , the payoff to Player  $i$ ,  $i = 1, 2$ , equals

$$\begin{aligned} M_y^i(Y^*) &:= L_y^i(Y^*; Y^*) = F_y^i(Y^*; Y^*) \\ &= D_{00}^i(y) + \nu_y(Y^*)[D_{11}^i(Y^*) - D_{00}^i(Y^*) - I^i]. \end{aligned} \quad (3)$$

Since symmetric strategies lead Players to stop simultaneously, such equilibria are referred to as *simultaneous stopping equilibria*. Note that  $M_y^i(\cdot)$  has a unique maximizer  $Y_M^i \geq Y_F^i$ . Its proof is trivial and, therefore, omitted.

**Proposition 2.** *For any  $Y^* \geq Y_F^j$ , such that  $M_y^i(Y^*) \geq L_y^i(Y_L^i; Y_F^j)$ , for all  $y \leq Y^*$ , it holds that  $(Y^*, Y^*)$  is a SPME.*

Note that existence of such joint stopping equilibria is not guaranteed in general. The sequential stopping equilibrium in proposition 1 always exists.

## 4 Endogenous Determination of the Leader and Follower Roles

It is often more realistic to assume that the roles of leader and follower are not exogenously determined, but the outcome of strategic interaction. In games with a first mover advantage players may try to preempt each other. Such preemptive situations arise whenever the value of becoming the leader exceeds the value of being the follower, while it is not optimal for either player to stop. Since the purpose of the paper is to investigate this competition and since any reasonable concept of equilibrium must have the follower stopping at  $Y_F^i$ , it will be implicitly assumed in the remainder that the follower's strategy is to stop at that threshold. Consequently, the second argument in (1) and (2) will be dropped for notational convenience. Furthermore, we use the notation  $L^i(y) := L_y^i(y)$ ,  $F^i(y) := F_y^i(y)$ , and  $M^i(y) := M_y^i(y)$ , for the instantaneous payoffs of becoming the leader, follower, and simultaneous stopping, respectively.

It can easily be seen that  $L_y^i(Y^*) \geq F_y^i(Y^*)$  iff  $L^i(Y^*) \geq F^i(Y^*)$ , for all  $y \in E$ . Since  $L^i(\cdot)$  and  $F^i(\cdot)$  are continuous, there exists  $Y_P^i < Y_L^i$  such that  $L^i(Y_P^i) = F^i(Y_P^i)$ . In fact, due to the monotonicity assumptions in Assumption 2,  $Y_P^i$  is unique. This point is called Player  $i$ 's *preemption point* and it is the lowest value for  $y$  at

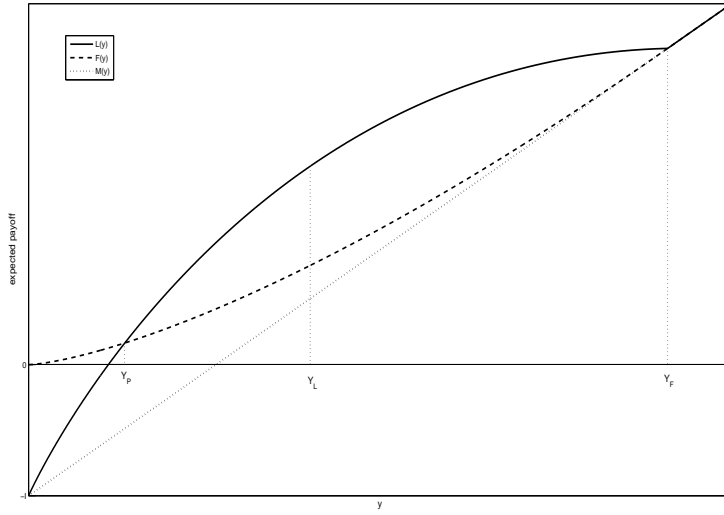


Figure 1: Value functions for leader, follower, and simultaneous stopping. The underlying stochastic process is a geometric Brownian motion.

which Player  $i$  would want to preempt Player  $j$ . Hence, the region in which Player  $i$  would wish to preempt Player  $j$  is  $S_P^i = [Y_P^i, Y_F^i)$ . Let the *preemption region* be defined as  $S_P := S_P^1 \cap S_P^2$ . We will focus on games with  $S_P \neq \emptyset$ . Obviously, Player  $i$  will only want to preempt Player  $j$  if there is a threat that Player  $j$  might preempt, i.e. when  $y \in S_P$ .

Combining this with the results from the previous section, the payoff structure of the game can be summarized as follows.

**Lemma 2.** *Under Assumptions 1–3 it holds that for every player  $i \in \{1, 2\}$  there exist unique thresholds*

1.  $Y_F^i \in \bar{E}$  such that  $M^i(y) = L^i(y) = F^i(y)$ , for all  $y \geq Y_F^i$ ;
2.  $Y_L^i < Y_F^i$  such that  $\max_{Y^*} L_y^i(Y^*) = L^i(y)$ , for all  $y \geq Y_L^i$ ;
3.  $Y_P^i < Y_L^i$  such that  $L^i(y) \geq F^i(y)$ , for all  $y \geq Y_P^i$ .

If one is unwilling to make the assumption that  $(Y_t)_{t \geq 0}$  exhibits no positive jumps, then one could always assume the validity of this lemma, without linking the payoffs to underlying fundamentals.<sup>3</sup> A plot of typical value functions is given in Figure 1.

<sup>3</sup>This is the approach that is taken in, for example, Thijssen et al. (2002).

## 4.1 Strategies and Payoffs

The main difference between deterministic timing games – where time is the state variable – and games where the state variable is a stochastic process is that in the latter case an agent’s strategy, however defined, can not be forward looking. In its most general form a strategy for Player  $i$  will be a process  $(X_t^i)_{t \geq 0}$  taking values in  $[0, 1]$ , where  $X_t^i$  is the probability with which Player  $i$  has stopped up to and including time  $t$ . Obviously,  $(X_t^i)_{t \geq 0}$  has to be a non-decreasing process. In addition, we want to rule out that players can act on information that has not yet been released. That is, “insider trading” should not be allowed. These two considerations lead us to conclude that  $(X_t^i)_{t \geq 0}$  must (i) be adapted to  $(\mathcal{F}_t)_{t \geq 0}$  and (ii) have càglàd (left-continuous with right-limits) sample paths. Due to the process  $(Y_t)_{t \geq 0}$  being a càdlàg semimartingale and  $(X_t^i)_{t \geq 0}$  being càglàd, stochastic integrals of the form  $\int X^i dY$  are well-defined (see Protter (2004)).

For our purposes it suffices to restrict attention to strategies that are driven by stopping times. Let  $\tau$  be a stopping time (relative to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ ). The *stopping strategy induced by the stopping time  $\tau$*  is given by

$$X^i(\tau) := \begin{cases} 0 & \text{if } t < \tau, \\ 1 & \text{if } t \geq \tau. \end{cases}$$

As already remarked, given the strong Markovian nature of  $(Y_t)_{t \geq 0}$ , the infinite time horizon and the no-positive-jump assumption, all optimal stopping problems in Section 2 take the form of trigger policies. Therefore, it stands to reason to focus on *threshold strategies*. These consist of a single threshold  $Y^i \in \bar{E}$ , with the convention that Player  $i$  stops at the induced first-hitting time  $\tau(Y^i)$ . So, the strategy space for each player is  $\bar{E}$ .

Let  $(Y^1, Y^2) \in \bar{E} \times \bar{E}$  and define  $\tau := \tau(Y^1) \wedge \tau(Y^2)$ . Then the expected payoff to Player  $i$  of the pair of thresholds  $(Y^i, Y^j)$  is given by

$$\begin{aligned} V_y^i(Y^i, Y^j) = & D_{00}^i(y) + \mathbb{E}^{P_y} \left[ e^{-r\tau} \left( 1_{[\tau(Y^i) < \tau(Y^j)]} L^i(Y_\tau) + 1_{[\tau(Y^i) > \tau(Y^j)]} F^i(Y_\tau) \right. \right. \\ & \left. \left. + 1_{[\tau(Y^i) = \tau(Y^j)]} W^i(Y_\tau) - D_{00}^i(Y_\tau) \right) \right], \end{aligned} \quad (4)$$

for all  $y \in E$ . Here  $W^i(\cdot)$  is a tie-breaking rule giving the expected payoff if both players stop at the same time. It will be elaborated on below.

Formally, a *stopping game* is a tuple  $\Gamma = (N, (\bar{E}, (V_y^i)_{y \in E}))_{i \in N}$ . A *preemption game* is a stopping game  $\Gamma$  with  $S_P \neq \emptyset$ . A *Markov equilibrium* in the stopping game  $\Gamma$  is a pair of thresholds  $(\bar{Y}^1, \bar{Y}^2) \in \bar{E} \times \bar{E}$ , such that  $V_y^i(\bar{Y}^i, \bar{Y}^j) \geq V_y^i(Y^i, \bar{Y}^j)$ , for all  $Y^i \in \bar{E}$ , all  $y \in E$ , and all  $i \in \{1, 2\}$ . A *preemption equilibrium* is a Markov equilibrium  $(\bar{Y}^1, \bar{Y}^2)$ , such that  $\bar{Y}^i < Y_L^i$ , for at least one  $i \in \{1, 2\}$ . For  $\varepsilon > 0$ ,

an  $\varepsilon$ -Markov equilibrium is a pair of strategies  $(\bar{Y}^1, \bar{Y}^2)$  such that for all  $y \in E$ , all  $i \in \{1, 2\}$ , and all  $Y^i \in \bar{E}$  it holds that

$$V_y^i(\bar{Y}^i, \bar{Y}^j) \geq V_y^i(Y^i, \bar{Y}^j) - \varepsilon.$$

A *preemption  $\varepsilon$ -equilibrium* is an  $\varepsilon$ -Markov equilibrium  $(\bar{Y}^1, \bar{Y}^2)$  such that  $\bar{Y}^i < Y_L^i$ , for at least one  $i \in \{1, 2\}$ .

The tie-breaking rule  $W^i(\cdot)$  deserves a bit more explanation. It is introduced here to allow for coordination between players if both wish to stop simultaneously. To allow for some generality, this function is assumed to be given by

$$W^i(y) = p_i(y)L^i(y) + p_j(y)F^i(y) + p_3(y)M^i(y), \quad \text{all } y \in E,$$

for some  $(p_1(y), p_2(y), p_3(y)) \geq 0$ , with  $p_1(y) + p_2(y) + p_3(y) = 1$ . In this set-up the probability with which Player  $i$  stops is  $p_i(y)$  and the probability that both stop simultaneously is  $p_3(y)$ . This formulation encompasses most contributions in the literature. For example, Murto (2004) does not allow coordination and, thus, assumes  $p_3(y) = 1$ , all  $y \in E$ ; Weeds (2002) assumes that  $p_1(y) = p_2(y) = 1/2$ ,  $y \in S_P$  and  $p_3(y) = 1$  otherwise; and Thijssen et al. (2002) argue, based on an argument by Fudenberg and Tirole (1985) for deterministic games, that  $p_3(y) = 0$ , all  $y \in S_P$ , but that  $p_1(y)$  and  $p_2(y)$  are such that  $W^i(y) = F^i(y)$ . A preemptive game which is such that  $W^i(y) = F^i(y)$ , all  $y \in S_P$ , is called a *rent-equalization* game. In Section 5 a non-cooperative defense of the rent-equalization property will be given. In the remainder of this section, we will mainly be concerned with comparing the effect of allowing for rent-equalization or not on the nature of equilibria in preemption games.

## 4.2 Preemption Equilibria

Throughout this subsection it will be assumed (wlog) that  $Y_L^1 \leq Y_L^2$ . The existence of equilibria depends crucially on the ordering of  $Y_L^1$  and  $Y_P^2$ , as well as the tie-breaking rule  $W^i(\cdot)$  in the preemption region. The following proposition establishes that a preemption game has no preemption equilibria if it does not have the rent-equalization property.

**Proposition 3.** *Let  $\Gamma$  be a preemption game satisfying Assumptions 1–3 with  $W^2(y) < F^2(y)$ , for all  $y \in S_P^2$ . Then no preemptive equilibrium exists.*

**Proof.** Note that a preemption equilibrium can only exist if  $Y_P^2 \leq Y_L^1$ .

Let  $y \in (Y_P^2, Y_L^1)$ . Suppose, by contradiction, that  $(\bar{Y}^1, \bar{Y}^2)$  is a preemption equilibrium. If  $y \geq \bar{Y}^i$ ,  $i = 1, 2$ , then both players stop simultaneously and  $V_y^2(\bar{Y}^2, \bar{Y}^1) = W^2(y) < F^2(y)$ , which implies that Player 2 wants to deviate.

If  $\bar{Y}^1 \leq y < \bar{Y}^2$ , then there exists  $\hat{Y}^1 \in (y, \bar{Y}^2 \wedge Y_L^1)$  such that  $L_y^1(\hat{Y}^1) > L_y^1(\bar{Y}^1) = L^1(y)$ . This holds because  $L_y^1(\cdot)$  is increasing on  $(y, \bar{Y}^2 \wedge Y_L^1)$ . So, Player 1 wishes to deviate. A similar reasoning applies to Player 2 if  $\bar{Y}^2 \leq y < \bar{Y}^1$ . ■

In Section 4.3 it will be shown that preemption games without the rent-equalization property sometimes allow for other equilibria.

If a preemption game satisfies the rent-equalization property, the picture looks very different.

**Proposition 4.** *Let  $\Gamma$  be a preemption game satisfying Assumptions 1–3 and the rent-equalization property.*

1. *Suppose that  $Y_P^2 \leq Y_L^1$ , and that  $Y_P^1 \neq Y_P^2$ . The following holds:*

- (a) *no preemption equilibrium exists;*
- (b) *if  $Y_P^1 < Y_P^2$ , then for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that all preemption  $\varepsilon$ -equilibria are of the form  $(Y^1, Y_P^2)$ , for any  $Y^1 \in [Y_P^2 - \delta, Y_P^2]$ ;*
- (c) *if  $Y_P^2 < Y_P^1$ , then for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that all preemption  $\varepsilon$ -equilibria are of the form  $(Y_P^1, Y^2)$ , for any  $Y^2 \in [Y_P^1 - \delta, Y_P^1]$ .*

2. *If  $Y_P^2 > Y_L^1$ , then no preemption equilibria exist.*

3. *If  $Y_P^1 = Y_P^2 \equiv Y_P$ , then  $(Y_P, Y_P)$  is the unique preemption equilibrium.*

**Proof.** First note that it is obvious that the statements are true for  $y \geq Y_F^1 \vee Y_F^2$ .

1. (a) A preemption equilibrium does not exist, since the best response of Player 1 to Player 2's strategy  $(Y_P^2)$  is not well-defined. After all, the function  $L_y^1(\cdot)$  is increasing on  $(a, Y_P^2)$ , so that it does not attain a maximum on this open set.
- (b) Suppose that  $Y_F^i \leq Y_F^j$ . Note that  $Y_P^2 < Y_F^i$ , since  $S_P \neq \emptyset$ . Let  $\varepsilon > 0$ . Take  $\delta > 0$  such that  $L_y^1(Y_P^2) - L_y^1(Y_P^2 - \delta) = \varepsilon$  and  $Y_P^2 - \delta \geq Y_P^1$ . Such a  $\delta$  exists because  $L_y^1(\cdot)$  is continuous and increasing on  $(a, Y_L^1]$ . Also, because of the strong Markovian nature of  $(Y_t)_{t \geq 0}$ ,  $\delta$  does not depend on  $y$ . Take any of the proposed equilibrium strategy pairs  $(\bar{Y}^1, \bar{Y}^2)$ . Consider the following cases.

i.  $y \in [Y_F^i, Y_F^j]$ .

In this case  $V_y^i(\bar{Y}^i, \bar{Y}^j) = L^i(y) = M^i(y) = F^i(y)$ , and  $V_y^j(\bar{Y}^j, \bar{Y}^i) = W^j(y) = F^j(y)$ . Player  $i$  has no incentive to deviate, since any deviation to  $\hat{Y}^i > y$  would lead to a payoff  $F^i(y)$ . The same holds for Player  $j$ .

ii.  $y \in [Y_P^2, Y_F^i]$ .

Note that  $V_y^k(\bar{Y}^k, \bar{Y}^\ell) = W^k(y) = F^k(y)$ , for  $k = 1, 2$ . So, for neither player would a deviation lead to a higher payoff.

iii.  $y < Y_P^2$ .

As in case (ii), Player 2 has no incentive to deviate to any  $\hat{Y}^2 > \bar{Y}^1$ . Let  $\hat{Y}^2 < \bar{Y}^1$ , Because of the no-positive-jump assumption it holds that

$$L_y^2(\hat{Y}^2) \leq L_y^2(\bar{Y}^1) \leq F_y^2(\bar{Y}^1),$$

where the first inequality holds because  $\hat{Y}^2 < \bar{Y}^1 < Y_L^2$  and  $L_y^2(\cdot)$  is non-decreasing on  $(a, Y_L^2)$ , and the second inequality holds by definition since  $\bar{Y}^1 < Y_P^2$ . Because  $\bar{Y}^1 < Y_L^1$  it holds that  $L_y^1(\hat{Y}^1) < L_y^1(\bar{Y}^1)$  for any  $\hat{Y}^1 < \bar{Y}^1$ . Since  $Y_L^1 \geq Y_P^2 > Y_P^1$ , it also holds that  $F_y^1(Y_P^2) < L_y^1(Y_P^2)$ . Therefore, all  $\varepsilon$ -best-responses to  $\bar{Y}^2 = Y_P^2$  are all in the interval  $(Y_P^2 - \delta, Y_P^2)$ . Since  $\bar{Y}^1 \in (Y_P^2 - \delta, Y_P^2)$ ,  $(\bar{Y}^1, \bar{Y}^2)$  constitutes an  $\varepsilon$ -Markov equilibrium.

As becomes clear from the above, any  $\varepsilon$ -Markov equilibrium must have  $\bar{Y}^2 \geq Y_P^2$ . However, there can be no preemption  $\varepsilon$ -equilibrium with  $Y_F^2 \geq \bar{Y}^2 > Y_P^2$ . For suppose there is. Then, Player 1's  $\varepsilon$ -best response will be to stop at some  $\hat{Y}^1 \in (\bar{Y}^2 - \delta, \bar{Y}^2)$ . For small enough  $\varepsilon$ , it will hold that  $\bar{Y}^2 - \delta > Y_P^2$ . In that case Player 2 will wish to deviate to  $\hat{Y}^2 = \bar{Y}^2 - \delta$ .

(c) The proof is analogous to that for the previous statement.

2. In this case it is dominant for Player  $i$  to wait until  $Y_L^i$  is hit. Consequently, she will never choose a strategy  $\bar{Y}^i < Y_L^i$ . Given this, Player  $j$  also has no incentive to choose a strategy  $\bar{Y}^j < Y_L^2$ .
3. Because of rent equalization, at  $Y_P$  each player is indifferent between stopping immediately and waiting. If one player chooses  $Y_P$ , then the other player has no incentive to deviate from  $Y_P$  because for all  $\hat{Y}^i < Y_P$ , it holds that

$$L_y^i(\hat{Y}^i) < F_y^i(\hat{Y}^i) \leq F_y^i(Y_P).$$

So,  $(Y_P, Y_P)$  is a preemption equilibrium.

Suppose, however, that there is another preemption equilibrium  $(\bar{Y}^1, \bar{Y}^2)$ , with, say,  $Y_P < \bar{Y}^1 \leq \bar{Y}^2$ . Let  $y \leq Y_P$ . Because of continuity of  $L_y^2(\cdot)$  and  $F_y^2(\cdot)$ , Player 2 can find  $\delta > 0$ , such that

$$\nu_{\bar{Y}^1 - \delta}(\bar{Y}^1)[F^2(\bar{Y}^1) - D_{00}^2(\bar{Y}^1)] < L^2(\bar{Y}^1 - \delta) - D_{00}^2(\bar{Y}^1 - \delta).$$

This, in turn, implies that  $L_y^2(\bar{Y}^1 - \delta) > F_y^2(\bar{Y}^1)$ , and, thus, that Player 2 should deviate.

■

### 4.3 Other Equilibria

Apart from preemption equilibria, preemption games may exhibit other equilibria. Again, it will be assumed throughout that  $Y_L^1 \leq Y_L^2$ .

#### 4.3.1 Sequential Equilibria

In this subsection we look at equilibria where players stop sequentially. The following lemma is trivial, but useful.

**Lemma 3.** *Let  $\Gamma$  be a preemption game satisfying Assumptions 1–3. Let  $Y_F := Y_F^1 \vee Y_F^2$ . If Player  $j$  plays a strategy  $Y^j \leq Y_F$ , then it is weakly dominant for Player  $i$  to choose a strategy  $Y^i \leq Y_F$ .*

**Proof.** Suppose that Player  $j$  plays  $Y^j \leq Y_F$  and that Player  $i$  chooses a strategy  $Y^i \in \bar{E}$ , with  $Y^i > Y_F^i$ . Let  $y \in [Y_F^i, Y^i)$ . Consider the following two cases.

1.  $Y^j \leq y$ .

In this case Player  $j$  becomes the leader and, therefore, Player  $i$ 's expected payoff is  $F^i(y)$ . Deviating to  $Y^i = Y_F^i$  would lead to the expected payoff  $M^i(y) = F^i(y)$ .

2.  $Y^j > y$ .

In this case Player  $i$  can become the leader and get an expected payoff  $L^i(y) > F^i(y)$  if  $y < Y_F^j$ . Otherwise, Player  $i$  gets the payoff  $M^i(y) = F^i(y)$ . ■

In light of this lemma we will only consider sequential Markov equilibria with  $Y^i \leq Y_F^i$  in this subsection. The sequential equilibrium from Proposition 1 survives in preemption games without the rent-equalization property as the following proposition shows.

**Proposition 5.** *Let  $\Gamma$  be a preemption stopping game satisfying Assumptions 1–3. Assume that the rent-equalization property does not hold. If  $Y_P^2 \geq Y_L^2$ , then  $(Y_L^1, Y_F^2)$  constitutes a Markov equilibrium.*

**Proof.** Consider the following cases.

1.  $y \geq Y_F^2$

In this region  $F^i(y) = M^i(y) = L^i(y)$ , for both players. Therefore, it is optimal for both to stop.

2.  $Y_L^1 \leq y < Y_F^2$

Given that Player 1 stops and  $W^2(y) < F^2(y)$ , Player 2 has no incentive to deviate. Conversely, given that Player 2 does not stop immediately it is optimal for Player 1

to stop.

### 3. $y < Y_L^1$

Given that Player 2 does not stop before  $Y_F^2$  is hit it is optimal for Player 1 to wait until  $Y_L^1$  is reached. Conversely, since  $Y_L^2 \geq Y_L^1$  and  $Y_F^2 \geq Y_L^1$  it holds for all  $y \leq \hat{Y} < Y_L^1$  that

$$\nu_{\hat{Y}}(Y_L^1)[F^2(Y_L^1) - D_{00}^2(Y_L^1)] \geq F^2(\hat{Y}) - D_{00}^2(\hat{Y}) > L^2(\hat{Y}) - D_{00}^2(\hat{Y}),$$

which, in turn, implies that  $L_y^2(\hat{Y}) < F_y^2(Y_L^1)$ . So, Player 2 prefers to become the follower at  $Y_L^1$  rather than to preempt and become leader at some  $y \leq \hat{Y} < Y_L^1$ . Finally, since  $W_y^2(Y_L^1) < F^2(Y_L^1)$ , it holds that  $F_y^2(Y_L^1) > W_y^2(Y_L^1)$ . Hence, Player 2 has no incentive to deviate to any  $y \leq \hat{Y} \leq Y_L^1$ . For any  $\hat{Y} > Y_L^1$ , it holds that  $V_y^2(\hat{Y}, Y_L^1) = V_y^2(Y_F^2, Y_L^1) = F_y^2(Y_L^1)$ . ■

In the above proposition, the condition that  $Y_F^2 \geq Y_L^1$  cannot be dispensed with. For suppose that  $Y_F^2 < Y_L^1$ . Then, since  $\nu_y(\cdot)$  is continuous, there exists  $\delta > 0$ , such that  $L_y^2(Y_L^1 - \delta) > F_y^2(Y_L^1)$ . So, Player 2 wishes to deviate and it follows from Proposition 3 that no sequential equilibrium exists.

In preemption games with the rent-equalization property, many more equilibria exist. However, they all lead to the same sequential scenario: Player 1 stops at  $Y_L^1$  and Player 2 at  $Y_F^2$ .

**Proposition 6.** *Let  $\Gamma$  be a preemption game satisfying Assumptions 1–3 and the rent-equalization property. If  $Y_F^2 > Y_L^1$ , then all preemption equilibria are of the form  $(Y_L^1, Y^2)$ , for any  $Y^2 \geq Y_F^2$ .*

**Proof.** It is dominant for Player 1 to stop whenever  $y \geq Y_L^1$ . Given that  $Y_F^2 > Y_L^1$ , it is weakly dominant for Player 2 not to preempt Player 1, since for any  $y \leq \hat{Y}^2 \leq Y_L^1$ , it holds that

$$L_y^2(\hat{Y}^2) \leq L_y^2(Y_L^1) \leq F_y^2(Y_L^1).$$

So, any Markov equilibrium  $(\bar{Y}^1, \bar{Y}^2)$  must have  $\bar{Y}^1 = Y_L^1$  and  $\bar{Y}^2 > Y_L^2$ . In addition, for any  $y \in [Y_L^1, Y_F^2)$  it is optimal for Player 2 to become follower rather than leader since  $W^2(y) < F^2(y)$ . So,  $\bar{Y}^2 \geq Y_F^2$ . For all  $Y_F^2 \leq y < Y_F^2$ , however, Player 2 is indifferent between stopping immediately and not stopping immediately because  $W^2(y) = F^2(y)$  due to rent equalization. So, any  $\bar{Y}^2 \in S_P$  leads to a Markov equilibrium. Every  $\bar{Y}^2 \geq Y_F^2$  trivially leads to a Markov equilibrium. ■

Again, the result depends crucially on the assumption that  $Y_F^2 > Y_L^1$ . If this is not the case then Proposition 4 gives preemption equilibria.



### 4.3.2 Simultaneous Equilibria

Apart from sequential equilibria, there may exist equilibria in which both players stop simultaneously. As is made clear in Section 4.2, such equilibria can never be preemptive. In fact, they only exist if the value of becoming the leader at any point in the preemption region is exceeded by the expected payoff of simultaneous stopping at some later date. Recall that, for  $Y^* > Y_F^i$ ,

$$M_y^i(Y^*) = D_{00}^i(y) + \nu_y(Y^*) [M^i(Y^*) - D_{00}^i(Y^*)], \quad (5)$$

where  $M^i(y) = D_{11}^i(y) - I^i$ . By a similar reasoning as in Lemma 1 it can easily be shown that (5) has a unique maximizer  $Y_M^i$ .

Using this notation the following equilibrium can exist in some cases.

**Proposition 7.** *Let  $\Gamma$  be a preemption game satisfying Assumptions 1–3. If  $Y^* > Y_F^1 \vee Y_F^2$  is such that*

$$L^i(y) \leq M_y^i(Y^*),$$

*for all  $y \in S_P^i$  and  $i \in \{1, 2\}$ , then  $(Y^*, Y^*)$  is a Markov equilibrium of  $\Gamma$ .*

**Proof.** Consider the following cases.

1.  $y > Y^*$ .

Given that layer  $j$  stops immediately, the best response of Player  $i$  is to stop immediately as well, since  $Y^* > Y_F^i$ .

2.  $y \leq Y^*$ .

Suppose that Player  $i$  deviates to  $\hat{Y}^i > Y^*$ . Then Player  $j$  will stop at time  $\tau(Y^*)$ . Since  $Y^* > Y_F^i$ , Player  $i$  will then stop immediately as well. So,  $V_y^i(\hat{Y}^i, Y^*) = V_y^i(Y^*, Y^*)$ . Conversely, if Player  $i$  deviates to  $Y_P^i \leq \hat{Y}^i < Y^*$ , then either

$$V_y^i(\hat{Y}^i, Y^*) = L^i(y) \leq M_y^i(Y^*) = V_y^i(Y^*, Y^*),$$

if  $y \geq \hat{Y}^i$ , or

$$\begin{aligned} V_y^i(\hat{Y}^i, Y^*) &= L_y^i(\hat{Y}^i) \\ &= D_{00}^i(y) + \nu_y(\hat{Y}^i) [L^i(\hat{Y}^i) - D_{00}^i(\hat{Y}^i)] \\ &\leq D_{00}^i(y) + \nu_y(\hat{Y}^i) [M^i(\hat{Y}^i, Y^*) - D_{00}^i(\hat{Y}^i)] \\ &= M^i(y, Y^*) = V_y^i(Y^*, Y^*), \end{aligned}$$

if  $y < \hat{Y}^i$ . ■

## 5 Non-Cooperative Coordination and Rent-Equalization

If one views the use of continuous time simply as a modeling tool that opens up the toolkit of stochastic calculus, then it is no great step to allow players to coordinate “in between two instantaneous points in time”. This idea can be formalized by using the definition of time as introduced in Dutta and Rustichini (1995). They view time as the two-dimensional set  $\mathcal{T} = \mathbb{R}_+ \times \mathbb{Z}_+$ , endowed with the lexicographic ordering, denoted by  $\geq_L$ , and the standard topology induced by  $\geq_L$ . That is, a typical time element is a pair  $s = (t, z) \in \mathcal{T}$ , which consists of a continuous and a discrete part. In the remainder,  $t$  refers to the continuous part and  $z$  to the discrete component. One can think of the continuous part of time as “process time” in which the stochastic environment evolves and the discrete part as “coordination time” in which players coordinate their actions. The great advantage of using this set-up is that it allows each part of the model to be analyzed in its most suitable way: stochastic evolution in continuous time and strategic interaction discrete time.

Obviously, the stochastic structure that has been used so far needs to be adapted to this new definition. Since we essentially want to keep the stochastic process  $(Y_t)_{t \geq 0}$  defined on the continuous part of time only, this is a fairly straightforward exercise. A filtration on  $(\Omega, \mathcal{F})$  is now a sequence of  $\sigma$ -fields,  $(\mathcal{F}_{(t,z)})_{(t,z) \geq_L (0,0)}$ , such that

$$\mathcal{F}_{(t,z)} \subseteq \mathcal{F}_{(t',z')} \subseteq \mathcal{F},$$

whenever  $(t, z) \leq_L (t', z')$ . For all  $y \in \mathbb{R}$ , let  $P_y$  be a probability measure on  $(\Omega, \mathcal{F})$  and define the process  $(Y_{(t,z)})_{(t,z) \geq_L (0,0)}$  such that  $Y_{(t,z)} = Y_t$ , for all  $t \in \mathbb{R}_+$  and  $z \in \mathbb{Z}_+$ . So, the extended process only moves in “process time” and is constant in “coordination time”. This way, stochastic integrals can also be extended trivially to operate on  $\mathcal{T}$ .

In this framework, the threshold strategies introduced in Section 4.1 are not so much the thresholds at which players stop, but the thresholds at which they are willing to engage in a coordination game. As argued by Fudenberg and Tirole (1985), this coordination game is most conveniently modeled as a “grab-the-dollar” game. This is an infinitely repeated game the stage game of which is as depicted in Figure 2. That is, play continues until at least one player “grabs the dollar”. We

|            | Grab             | Don't grab       |
|------------|------------------|------------------|
| Grab       | $M^1(y), M^2(y)$ | $L^1(y), F^2(y)$ |
| Don't grab | $F^1(y), L^2(y)$ | play again       |

Figure 2: The coordination game.

allow for mixed strategies in the stage game. Given that the “grab-the-dollar” game

is (potentially) infinitely repeated we can restrict attention to stationary strategies and denote the probability with which Player  $i$  grabs the dollar in the stage game by  $\alpha^i$ .

The payoff to Player  $i$  in this repeated game depends on the probability that she is the first to grab the dollar. For a given pair  $(\alpha^1, \alpha^2)$ , the probability that Player  $i$  grabs the dollar first is denoted by  $p_i(y)$  and is equal to

$$\begin{aligned} p_i(y) &= \alpha^i(1 - \alpha^j) + \alpha^i(1 - \alpha^i)(1 - \alpha^j)^2 + \dots \\ &= \alpha^i \sum_{z=1}^{\infty} (1 - \alpha^i)^{z-1} (1 - \alpha^j)^z \\ &= \frac{\alpha^i(1 - \alpha^j)}{\alpha^i + \alpha^j - \alpha^i \alpha^j}. \end{aligned} \tag{6}$$

Similar computations show that the probabilities that Player  $j$  grabs the dollar first, denoted by  $p_j(y)$ , and that both players grab the dollar simultaneously, denoted by  $p_3(y)$ , are equal to

$$p_j(y) = \frac{\alpha^j(1 - \alpha^i)}{\alpha^i + \alpha^j - \alpha^i \alpha^j}, \quad \text{and} \quad p_3(y) = \frac{\alpha^i \alpha^j}{\alpha^i + \alpha^j - \alpha^i \alpha^j}, \tag{7}$$

respectively.

The expected payoff to Player  $i$  in the repeated game then equals

$$W_y^i(\alpha^i, \alpha^j) = p_i(y)L^i(y) + p_j(y)F^i(y) + p_3(y)M^i(y). \tag{8}$$

It is obvious that it is a weakly dominant strategy to set  $\alpha^i = 1$  whenever  $y \geq Y_F^i$  and  $\alpha^i = 0$ , whenever  $y < Y_P^i$ . Furthermore, it is easy to see that, for each  $y \in S_P$ , there is a unique mixed strategy equilibrium where

$$\bar{\alpha}^i = \frac{L^j(y) - F^j(y)}{L^j(y) - M^j(y)}. \tag{9}$$

The expected payoffs in this equilibrium are easily confirmed to be  $W_y^i(\bar{\alpha}^i, \bar{\alpha}^j) = F^i(y)$ . In other words, non-cooperative coordination through a “grab-the-dollar” game leads to rent-equalization. This gives, therefore, a non-cooperative justification for the assumption that  $W^i(y) = F^i(y)$  for  $y \in S_P$ .

## 6 Examples

In this section some examples are given that illustrate the applicability of the results derived so far. Attention is mainly focussed on preemption equilibria in games that satisfy the rent-equalization property.

## 6.1 Spectrally Negative Lévy Processes

A *Lévy process* is an adapted process with independent and stationary increments.<sup>4</sup> Each Lévy process has a càdlàg version, which is the one we will work with. For Borel sets  $U$  with  $0 \notin \bar{U}$ , the *Poisson random measure* of  $(Y_t)_{t \geq 0}$  is given by  $N(t, U) := \sum_{0 < s \leq t} 1_U(\Delta Y_s)$ . So,  $N(t, U)$  is the number of jumps with size in  $U$ . The corresponding compensated Poisson random measure is denoted by  $\tilde{N}(t, U)$ . The intensity of the Poisson process is denoted by  $\lambda$  and the Lévy measure is defined as  $m(U) := \mathbb{E}_y[N(1, U)]$ . In differential form a time homogeneous Lévy process can be written as

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t + \int_{\mathbb{R}} \gamma(Y_t, z)\tilde{N}(dz, dt), \quad (10)$$

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion. Assume that  $(Y_t)_{t \geq 0}$  takes values in  $E = (a, b)$  and that

$$a - y < \gamma(y, z) \leq 0, \quad \text{for all } (y, z) \in E \times \mathbb{R}. \quad (11)$$

This assumption ensures that  $(Y_t)_{t \geq 0}$  has no upwards jumps. A Lévy process that satisfies (11) is called *spectrally negative*. Such processes are useful to model situations where “success comes on foot and leaves on horseback”.

The generator of the process  $(Y_t)_{t \geq 0}$ , killed at rate  $r > 0$ , as defined for  $f \in C^2$ , is given by

$$\begin{aligned} \mathcal{L}_Y(g) = & \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y) \\ & + \lambda \int_{\mathbb{R}} [g(y + \gamma(y, z)) - g(y) - g'(y)\gamma(y, z)] m(dz) - rg(y). \end{aligned} \quad (12)$$

Suppose that (12) has an increasing  $C^2$  solution  $g$ , such that  $g(a) = 0$ . Take  $Y^* \geq y$ . Then Dynkin's formula gives

$$\begin{aligned} \mathbb{E}_y \left[ e^{-r\tau(Y^*)} g(Y_{\tau(Y^*)}) \right] &= g(y) + \mathbb{E}_y \left[ \int_0^{\tau(Y^*)} \mathcal{L}_Y g(Y_t) dt \right] \\ \iff \mathbb{E}_y \left[ e^{-r\tau(Y^*)} \right] &= \frac{g(y)}{g(Y^*)}. \end{aligned}$$

All diffusions, i.e. Lévy processes without jumps, are spectrally negative. For several well-known classes of spectrally negative Lévy processes the stochastic discount factor  $\nu_y(\cdot)$  can be computed explicitly. First consider an arithmetic Brownian motion (ABM). This is a Lévy process the evolution of which is described by the stochastic differential equation (10) with  $\mu(y) = \mu \in \mathbb{R}$ ,  $\sigma(y) = \sigma > 0$ , and  $\lambda = 0$ . For this process it holds that

$$\nu_y(Y^*) = e^{\beta_1(y - Y^*)},$$

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<sup>4</sup>See, for example, Øksendal and Sulem (2007).

where  $\beta_1 > 0$  is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 + \mu\beta - r = 0.$$

A second example is the geometric Brownian motion (GBM), which takes  $\mu(y) = \mu y$ ,  $\sigma(y) = \sigma y$ ,  $\lambda = 0$  in (10). In addition it is assumed that  $r > \mu$ . It then holds that

$$\nu_y(Y^*) = \left(\frac{y}{Y^*}\right)^{\beta_1},$$

where  $\beta_1 > 1$  is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0.$$

If one adds Beta distributed negative jumps to a GBM, i.e.  $\lambda > 0$  and

$$m'(z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1}, \quad a, b > 0,$$

where  $\Gamma(\cdot)$  is the Gamma function, then it follows that (see Alvarez and Rakkolainen (2010))

$$\nu_y(Y^*) = \left(\frac{y}{Y^*}\right)^{\beta_1},$$

where  $\beta_1 > 0$  is the positive root of the equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \left(\mu + \frac{\lambda a}{a+b}\right)\beta - (r + \lambda) + \lambda \frac{\Gamma(a+b)\Gamma(b+\beta)}{\Gamma(b)\Gamma(a+b+\beta)} = 0.$$

The results in this paper also apply to processes that exhibit mean-reversion. Consider, for example, the diffusion

$$\frac{dY}{Y} = \eta(\bar{Y} - Y)dt + \sigma dW,$$

on  $\mathbb{R}_+$ , where  $\bar{Y}$  is the long-run value of  $Y$  and  $\nu$  determines the speed of mean-reversion. The generator of this process is

$$\mathcal{L}_Y g = \frac{1}{2}\sigma^2 Y^2 g'' + \eta(\bar{Y} - Y)g' - rg.$$

The solution to  $\mathcal{L}_Y g = 0$  is (see, for example, Dixit and Pindyck (1994))

$$g(y) = Ay^{\theta_1} H\left(\frac{2\eta}{\sigma^2}y; \theta_1, b(\theta_1)\right) + By^{\theta_2} H\left(\frac{2\eta}{\sigma^2}y; \theta_2, b(\theta_2)\right),$$

where

$$H(x; \theta, b) = \sum_{n=0}^{\infty} \frac{\Gamma(\theta + n)/\Gamma(\theta)}{\Gamma(b + n)/\Gamma(b)} \frac{x^n}{n!},$$

is the generalized hypergeometric function,  $\theta_1 > 0$  and  $\theta_2 < 0$  are the roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta(\theta-1) + \eta\bar{Y}\theta - r = 0,$$

and  $b(\theta) = 2(\theta + (\eta/\sigma^2)\bar{Y})$ . Since we are looking for solutions with  $g(0) = 0$ , it must hold that  $B = 0$ . Since  $g$  is increasing if  $A > 0$ , we get that

$$\nu_y(Y^*) = \left(\frac{y}{Y^*}\right)^{\theta_1} \frac{H\left(\frac{2\eta}{\sigma^2}y; \theta_1, b(\theta_1)\right)}{H\left(\frac{2\eta}{\sigma^2}Y^*; \theta_1, b(\theta_1)\right)}.$$

These result can be used to find the optimal thresholds  $Y_F^i$  and  $Y_L^i$ . For  $k = 0, 1$ , namely, these thresholds are obtained as the solution to the optimization problem

$$\max_{Y^*} \nu_y(Y^*) [D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i],$$

which leads to the first order condition (assuming differentiability of  $D_{k\ell}^i(\cdot)$ ):

$$\frac{\partial \nu_y(Y^*)}{\partial Y^*} [D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i] + \nu_y(Y^*) \frac{\partial}{\partial Y^*} [D_{1k}^i(Y^*) - D_{0k}^i(Y^*)] = 0. \quad (13)$$

For ABM, GBM with beta distributed negative jumps, and mean-reversion we have

$$\begin{aligned} \frac{\partial \nu_y(Y^*)}{\partial Y^*} &= -\beta_1 \nu_y(Y^*), \quad \frac{\partial \nu_y(Y^*)}{\partial Y^*} = -\frac{\beta_1}{Y^*} \nu_y(Y^*), \quad \text{and} \\ \frac{\partial \nu_y(Y^*)}{\partial Y^*} &= \frac{2\eta}{\sigma^2} \frac{\theta}{b} H\left(\frac{2\eta}{\sigma^2}Y^*; \theta_1 + 1, b(\theta_1) + 1\right), \end{aligned}$$

respectively. Note that this implies that in all these cases (13) does not depend on  $y$  and that, thus,  $Y^*$  is a constant, as expected.

For several different stochastic processes the preemption region is plotted in Figure 3 as a function of volatility. The net present values are taken to be linear in the underlying shock:  $D_{k\ell}^i(y) = D_{k\ell}y$ , where  $D_{k\ell}$  are constants such that  $D_{10} > D_{11} \geq D_{00} \geq D_{01}$  and  $D_{10} - D_{00} > D_{11} - D_{01}$ . As can be easily seen, the preemption region tends to get wider in the case of higher volatility. This happens because a higher volatility does not influence the present values whereas it increases option values. These option values have a bigger impact on the follower threshold than on the preemption threshold. After all, preemptive pressure erodes the option value for the leader. This implies that in games with higher levels on uncertainty, it is more likely that a preemptive situation occurs.

## 6.2 Industry Investment and Welfare

Many papers on preemption games deal with firms investing in a new product, or a new technology. An important question in such models is what the impact of

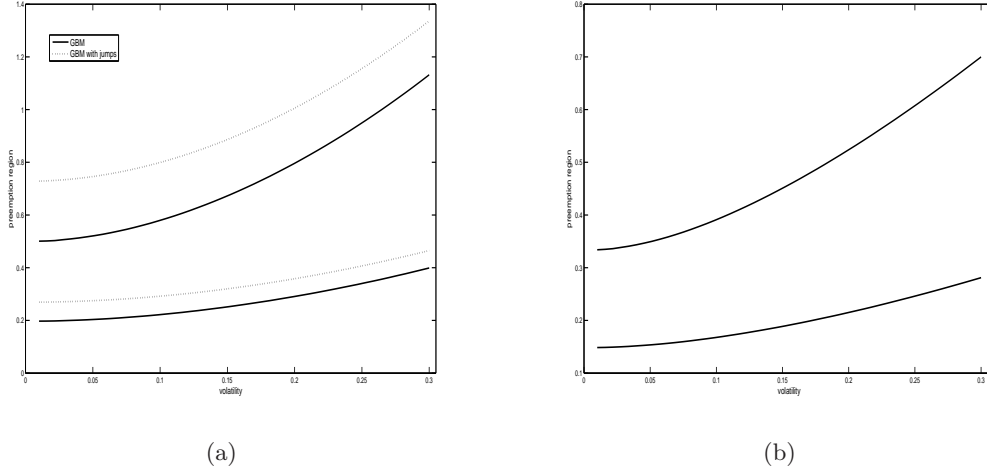


Figure 3: Preemption region as function of volatility  $\sigma$  for different stochastic processes. General parameter values are  $I = 1$ ,  $r = .1$ ,  $D_{10} = 10$ ,  $D_{11} = 3$ ,  $D_{00} = 2$ , and  $D_{01} = 1$ . (a) GBM (solid lines) and GBM with negative Beta jumps (dashed lines) with  $\mu = .03$ ,  $\lambda = .1$ ,  $a = 1.5$ ,  $b = 2$ ; (b) exponential mean-reverting with  $\bar{Y} = 2$  and  $\eta = .015$ .

preemptive behaviour on welfare is. In general, it is difficult to say anything about this, but here we consider a fairly straightforward example. Let inverse demand in the industry be given by  $D(Q) = Y - Q$ , where  $Y$  is a stochastic shift variable, which follows the GBM

$$\frac{dY}{Y} = \mu dt + \sigma dW.$$

For simplicity it is assumed that there are no costs of production beyond the sunk costs of investment  $I > 0$ , and that profits are discounted at a rate  $r > 2\mu + \sigma^2$ . We assume the following present values

$$\begin{aligned} D_{10}^i(y) &= \mathbb{E}_y \left[ \int_0^\infty e^{-rt} \max_{Q_t} (Y_t - Q_t) Q_t \right] = \frac{1}{4} \frac{y^2}{r - 2\mu - \sigma^2}, \\ D_{11}^i(y) &= \mathbb{E}_y \left[ \int_0^\infty e^{-rt} \max_{q_t^i} (Y_t - q_t^i - q_t^j) q_t^i \right] = \frac{1}{9} \frac{y^2}{r - 2\mu - \sigma^2}, \\ D_{00}^i(y) &= D_{01}^i(y) = 0. \end{aligned}$$

Note that the stochastic process of interest is  $(X_t)_{t \geq 0}$ , where  $X_t = Y_t^2$ . A straightforward application of Ito's lemma shows that  $(X_t)_{t \geq 0}$  follows the GBM

$$\frac{dX}{X} = (2\mu + \sigma^2)dt + 2\sigma dW.$$

Let

$$X^* = \frac{\beta_1}{\beta_1 - 1} (r - 2\mu - \sigma^2) I,$$

where  $\beta_1 > 1$  is the positive root of the quadratic equation  $\sigma^2\beta(\beta-1) + (2\mu + \sigma^2)\beta - r = 0$ . The leader and follower thresholds are given by

$$X_L = 4X^*, \quad \text{and} \quad X_F = 9X^*,$$

respectively. Consumer surplus in the case where  $k = 0, 1, 2$  firms have invested is denoted by  $CS_k(\cdot)$ , and equals

$$\begin{aligned} CS_0(x) &= 0, \\ CS_1(x) &= \mathbb{E}_y \left[ \int_0^\infty e^{-rt} \frac{1}{2} \left( Y_t - \frac{1}{2} Y_t \right) \frac{1}{2} Y_t dt \right] = \frac{1}{8} \frac{y^2}{r - 2\mu - \sigma^2}, \quad \text{and} \\ CS_2(x) &= \mathbb{E}_y \left[ \int_0^\infty e^{-rt} \frac{1}{2} \left( Y_t - \frac{1}{3} Y_t \right) \frac{2}{3} Y_t dt \right] = \frac{2}{9} \frac{y^2}{r - 2\mu - \sigma^2}. \end{aligned}$$

Note that

$$\begin{aligned} W_1(x) &:= D_{10}^i(y) + C_1(y) = \frac{3}{8} \frac{x}{r - 2\mu - \sigma^2}, \quad \text{and} \\ W_2(x) &:= 2D_{11}^i(y) + C_2(y) = \frac{4}{9} \frac{x}{r - 2\mu - \sigma^2}. \end{aligned}$$

A welfare maximizing social planner would choose the thresholds for the first and second firm to invest such that  $X^1$  and  $X^2$  solve

$$\max_{X^1, X^2 \geq X^1} \{ \nu_x(X^1)[W_1(X^1) - I] + \nu_x(X^2)[W_2(X^2) - W_1(X^2) - I] \}.$$

Solving this problem gives

$$X^1 = \frac{8}{3}X^*, \quad \text{and} \quad X^2 = \frac{72}{5}X^*,$$

respectively. Note that  $X^2 > X_F > X_L > X^1$ . So, a social planner would have the first firm invest sooner than in the case of exogenously determined firm roles, but would have the second firm invest later. Since  $L(X^1) \geq F(X^1)$ , no general statement can be made about whether preemption leads to investment before the social optimum. For various values of the volatility parameter,  $\sigma$ , the welfare loss for both cases is plotted in Figure 4. As can be seen, the thresholds are (as expected) increasing in volatility. Also, for small values of  $\sigma$ , preemptive investment actually takes place later than the social optimum. The welfare loss (as % of the social optimum) is higher when firm roles are exogenous. As volatility increase, the welfare loss in both cases reduces. This is mainly driven by the fact that  $2\mu + \sigma^2 \uparrow r$ , so that the present values and, hence, welfare levels diverge.

In order to disentangle the option and present value effects on welfare results, consider the following slight modification. Rather than letting  $Y$  denote a stochastic



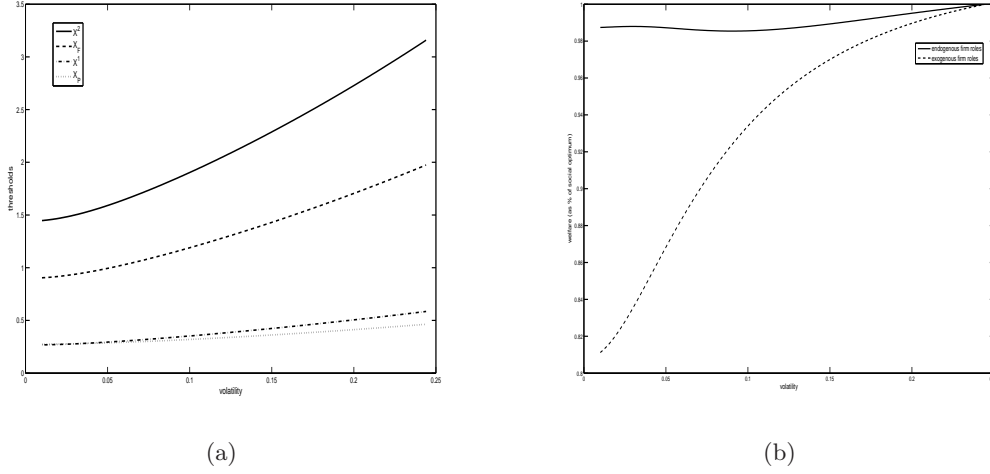


Figure 4: Welfare analysis driven by GBM with  $\mu = .02$ ,  $r = .1$ , and  $I = 1$ : (a) thresholds as a function of volatility  $\sigma$ , (b) welfare as a % of the social optimum for the case with exogenous firm roles (dashed line) and endogenous firm roles (solid line).

shift variable in the inverse demand function, suppose that  $P(Q) = YD(Q)$ , where  $D(Q) = 1 - Q$ . This formulation makes all present values independent of  $\sigma$  and ensures similar results as above. Let

$$Y^* = \frac{\beta_1}{\beta_1 - 1}(r - \mu)I,$$

where  $\beta_1 > 1$  is the positive root of the quadratic equation  $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ . The leader and follower thresholds are given by

$$Y_L = 4Y^*, \quad \text{and} \quad Y_F = 9Y^*,$$

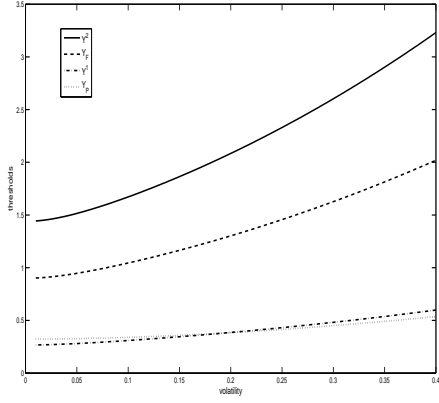
respectively. Consumer surplus equals  $CS_0(y) = 0$ ,

$$CS_1(y) = \frac{1}{8} \frac{y}{r - \mu}, \quad \text{and} \quad CS_2(y) = \frac{2}{9} \frac{y}{r - \mu},$$

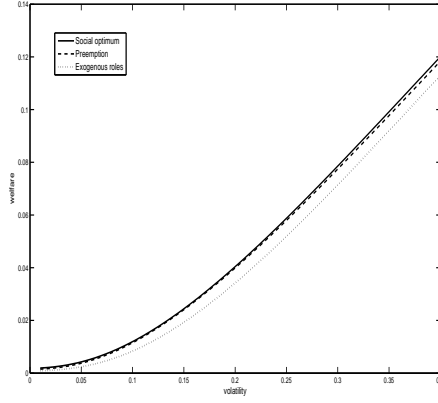
so that

$$W_1(y) = \frac{3}{8} \frac{y}{r - \mu}, \quad \text{and} \quad W_2(y) = \frac{4}{9} \frac{y}{r - \mu}.$$

For similar parameter values as in Figure 4, the plots in Figure 5 give the thresholds and (absolute) welfare levels of the exogenous and endogenous firm role cases. Note that even for somewhat higher values of  $\sigma$ , the preemption threshold is above the social optimum. So, even though preemption leads to rent equalization, those rents are still too high to induce firms to invest at the social optimum. Also, for higher values of  $\sigma$  welfare levels are actually diverging. This implies that, unlike in the



(a)



(b)

Figure 5: Welfare analysis driven by GBM with  $\mu = .02$ ,  $r = .1$ , and  $I = 1$ : (a) thresholds as a function of volatility  $\sigma$ , (b) welfare of the social optimum (solid line), exogenous firm roles (dashed line) and endogenous firm roles (dotted line) cases.

previous case, higher levels of uncertainty actually lead to behaviour that is further removed from the social optimum. Finally, welfare levels are actually increasing in volatility. This is due to a balance of a present value and a stochastic discount effect. All welfare functions contain components of the form  $(y/Y^*)^{\beta_1} F(Y^*)$ . The present value component  $F(Y^*)$  is homogeneous of degree one, whereas the stochastic discount factor is of degree  $1/\beta_1 \in (0, 1)$  (in  $Y^*$ ). Therefore, an increase in  $Y^*$  due to, for example, higher volatility, has a bigger upward effect on the present value than it has a negative effect on the discount factor.

## 7 Conclusion

This paper presents equilibrium results for a large class of timing games in which there is a first-mover advantage. The approach taken here exploits the Markovian nature of many classes of often used stochastic processes. This allows for a more straightforward analysis than in most existing contributions. In addition, the issue of rent-equalization has been studied separately and embedded in the timing game, which makes it easier to see how it arises and whether its presence is reasonable.

In some applications where the environment changes very rapidly – such as financial markets for high frequency traded products – it might not be. In that case, no preemption equilibria exist and only collusive simultaneous stopping equilibria or asymmetric sequential equilibria can be obtained. In fact, no symmetric equilibria may exist at all. A problem with the asymmetric equilibria arises in

symmetric games. In such games there are two asymmetric equilibria  $(Y_L, Y_F)$  and  $(Y_F, Y_L)$  and it is not clear, *a priori*, which one should be selected. In either case, though, the leader and follower roles are determined later than in the case where rent-equalization is possible. Therefore, somewhat contradictory, in fast moving situations, stopping gets delayed.

The welfare analysis in Section 6.2 suggests that, in industry investment situations, preemption is actually a good thing as it gets us closer to the social optimum than asymmetric sequential equilibria do. Therefore, the availability of time for coordination (i.e. playing of the “grab-the-dollar” game) is socially desirable.

Another advantage of the set-up in this paper is that the ideas can easily be adapted to games in which there is a second mover advantage (wars of attrition). Also, games with both a first mover advantage on the upside and a second mover advantage on the downside can be analyzed using this framework. In particular, this opens up the possibility of a proper analysis of the investment and disinvestment behavior of competing firms.

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